The topological index of Atiyah-Singer.

X compact manifold. By the Whitney embedding theorem can assume that $X \subset \mathbb{R}^n$.

This induces an embedding $TX \subset T\mathbb{R}^n \cong \mathbb{R}^{2n}$.

Take the tubular neighborhood of TX in $\mathbb{R}^{2n}$.

It is diffeomorphic to the total space of the normal bundle N over X. N is canonically K-oriented by an appropriate Clifford symbol. The one obtains the composition

$$K^*(TX) \xrightarrow[\text{iso}]{\text{Thom}} K^*(N) \hookrightarrow K^*(\mathbb{R}^{2n}) \xrightarrow[\text{periodicity}]{\text{Bott}} K^*(\cdot) = \mathbb{Z}$$

It is the topological index map of Atiyah-Singer.

Applications of NCG (to other fields)

1) The Novikov Conjecture.

G a discrete group, $BG = K(G,1)$ (the classifying space of G = the Eilenberg-MacLane space,

$[X, BG]$ = isoclasses of regular G-Galois coverings of X,

$$\pi_*(K(G,1)) \cong \begin{cases} G & * = 1 \\ 0 & * \neq 1 \end{cases}\quad).$$

$f : X \xrightarrow[\text{continuous}]{} BG$, X compact oriented n-fold,

$L_i(X) \in H^{4i}(X, \mathbb{Z})$ i-th Hirzebruch polynomial in Pontryagin classes

$x \in H^{n-4i}(X, \mathbb{Z})$

Conjecture. $\langle f^* x \cup L_i(X), [X] \rangle \in \mathbb{Q}$ is an invariant of the oriented homotopy type of X.

2) The positive scalar curvature problem on Spin-closed manifolds.

D Dirac operator, $\mathrm{ind}(D^+) = \langle \hat{A}(X), [X] \rangle$.

Theorem. [Lichnerowicz] If X admits a Riemannian metric with positive scalar curvature, then $\hat{A}(X) = 0$.

<u>Hint</u> $D^2 = \nabla^* \nabla + \frac{1}{4} S$, S scalar curvature, ∇ Levi-Civita connection on spinors.

$S > 0 \Rightarrow \ker D = 0 \Rightarrow \mathrm{ind}(D^+) = \dim \ker D^+ - \dim \ker D^-$
$= 0$.

3) **The Baum-Connes Conjecture.**

G loc. cpct group. X a proper G-space

(i.e. $\{g \in G \mid g(K) \cap L \neq \emptyset\}$ is compact for all compacts $K, L \subset X$.)

A, B C_0-section algebras of continuous fields of C^*-algebras over X with G-action.

(e.g. $C_0(X)$, $C_0(Cl(TX))$, $C_0(X, D)$, ...)

The groupoid-equivariant KK-theory $KK^{G \ltimes X}(B, A)$ has cycles of type of "continuous" families $\{(\mathcal{H}_x, F_x, \varphi_x) \mid x \in X\}$ parametrized by X, where $(\mathcal{H}_x, F_x, \varphi_x)$ is a cycle for $KK(B_x, A_x)$.

The equivariance condition: $g : H_x \to H_{gx}$,

$g F_x g^{-1} - F_{gx} \in \mathcal{K}(H_x)$

Example. $A = B = C_0(X)$

$$KK_*^{G \ltimes X}(C_0(X), C_0(X)) =: RK_G^*(X)$$

The representable G-equivariant K-theory of X,

$RK_G^0(X) = [X, \mathcal{F}]$, $\mathcal{F}$ space of Fredholm operators on $L^2(G) \otimes \ell^2(\mathbb{N})$.

Definition. EG is a free G-space which is contractible.

Remark. EG is determined uniquely up to G-homotopy.

Excercise 42. Let V be a free G-space and X be a G-space. Show that $V \times X$ with a diagonal G-action is a free G-space.

Solution. $gv = v \Rightarrow g = e$

$$g(v, x) = (gv, gx) = (v, x) \Longrightarrow gv = v \Rightarrow g = e.$$

Example. If X and G are compact, then

$$RK^*_G(X) \cong K^*_G(X) := K^*(EG \times^G X)$$

Definition. $\underline{E}G$ is a G-proper space s.t.

$\forall_X$ proper G-CW-complex $\quad [X, \underline{E}G]_G = \{*\}$)

G-homotopy classes

Remark. $\underline{E}G$ unique up to G-homotopy.
It is called a *universal proper G-space*.

Proposition. Let G be a countable discrete group. There exists a universal proper G-space.

Proof. (Sketch of) [construction of Kasparov-Skandalis] Let $\mathcal{M}_t(G)$ be a space of measures on G of mass $\leq t$. It is a compact space in the topology of pointwise convergence. Then

$$\underline{E}G \overset{G}{\sim} \mathcal{M}_1(G) \setminus \mathcal{M}_{1/2}(G).$$

Proposition. Let M be a complete and simply connected Riemannian manifold of nonpositive sectional curvature. If a group G acts properly and isometrically on M then

$$\underline{E}G \overset{G}{\sim} M.$$

Example. G semisimple Lie group, $M = G/K$ a symmetric G-space (i.e. $\exists\, \sigma$ an involution on G s.t. $K = G^{\sigma}$).

Example. Let G be any Lie group with finitely many connected components. Up to conjugacy G has a unique maximal compact subgroup K. Then

$$\underline{E}G = G/K.$$

Example. G any locally compact group with a compact component group G/G_0. Up to conjugacy G has a unique maximal compact subgroup K. Then

$$\underline{E}G = G/K.$$

... and more: p-adic reductive, adelic, hyperbolic groups etc.

Example. Let G be a topological group without nontrivial compact subgroups. Then

$$\underline{E}G = EG.$$

Conjecture. [Baum–Connes] For any G-C^*-algebras A, B the natural map

$$p^*_{\underline{E}G} : KK^G(B, A) \to KK^{G \ltimes \underline{E}G}(C_0(\underline{E}G, B), C_0(\underline{E}G, A)),$$

where $\underline{E}G$ is the universal proper G-space,

$$p^*_{\underline{E}G}((\mathcal{H}, F, \varphi)) = [\{(\mathcal{H}_x = \mathcal{H}, F_x = F, \varphi_x = \varphi)\} \mid x \in \underline{E}G\}],$$

is an isomorphism for "many cases".

Remark. It is not an original formulation. In general it is false. In many cases it is true.

Example. $G = F_2$. It acts freely on a tree T. Take $H = \ell^2(T^{\text{vertex}}) \oplus \ell^2(T^{\text{edge}}) = H^+ \oplus H^-$ with a $\mathbb{Z}/2\mathbb{Z}$-grading. Define an operator $H^+ \to H^-$

$$b(\delta_v) = \delta_{s(v)}$$

where v is a vertex having an edge $s(v)$ adjacent to v that leads to the base point, and

$$b(\delta_{\text{base pt}}) = 0.$$

then: b is Fredholm of index $\underline{1}$,

$\forall g \in F_2$ $\quad gbg^{-1} - b$ is compact (in fact of finite rank),

$\underline{E}G = |T|$, the underlying topological space of T.

Theorem. For $G = F_2$ the map $P^*_{\underline{E}G}$ is invertible.

Proof. (Idea of) F from the polar decomposition of b satisfies $\gamma := [(H, F)] = \underline{1} \in KK^G(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$,

and $P^*_{\underline{E}G}(\gamma) = 1 \in KK^{G \ltimes \underline{E}G}(C_0(\underline{E}F_2), C_0(\underline{E}F_2)) \cong \mathbb{Z}$)

$\square$

Corollary. Let A be a F_2-C^*-algebra that satisfies the UCT. Then so does $A \rtimes_r F_2$.

Proof. (Sketch of) $\gamma = \alpha \otimes_P \beta$, $\alpha \in KK^G(\mathbb{C}, P)$, $\beta \in KK^G(P, \mathbb{C})$, P a proper G-C^*-algebra. Then $P \rtimes G$ satisfies UCT. $(P \otimes A) \rtimes G$ satisfies UCT for all A. Then, $\alpha \otimes 1_A$ is invertible in $KK^G(A, A \otimes P)$, so is its image under the descent map $KK^G(A, A \otimes P) \to KK(A \rtimes G, (A \otimes P) \rtimes G)$. But $A \rtimes G \overset{KK}{\sim} (A \otimes P) \rtimes G$. $\square$

Example. $\Gamma < SL(2,\mathbb{R})$ s.t. $SL(2,\mathbb{R})/\Gamma$ cpct. $\underline{E}\Gamma = \mathbb{H}^2$ (hyperbolic upper-plane). There is also $\gamma = [(H,F)]$ a generator of $KK^G(\mathbb{C},\mathbb{C}) \simeq \mathbb{Z}$ [Higson-Kasparov].

$\partial\mathbb{H}^2 := \mathbb{R}\cup\{\infty\}$ (hyperbolic boundary). Γ acts on $\partial\mathbb{H}^2$ isometrically, $C(\partial\mathbb{H}^2)\rtimes\Gamma$ is purely infinite and nuclear (i.e. each of its non zero hereditary sub C^*-algebras contains an infinite projection and the identity as a c.p. map approximately factors through matrix algebras)

"NC partition of unity"

$B \subset A$
$0 \leq a \leq b$
$\Downarrow$
$a \in B$

Murray-von Neumann equivalent to its subprojection

How to construct a Dirac class in $K^0(C(\partial \mathbb{H}^2) \rtimes \Gamma)$?

There is no an obvious candidate for a Dirac operator. Baum - Connes helps.

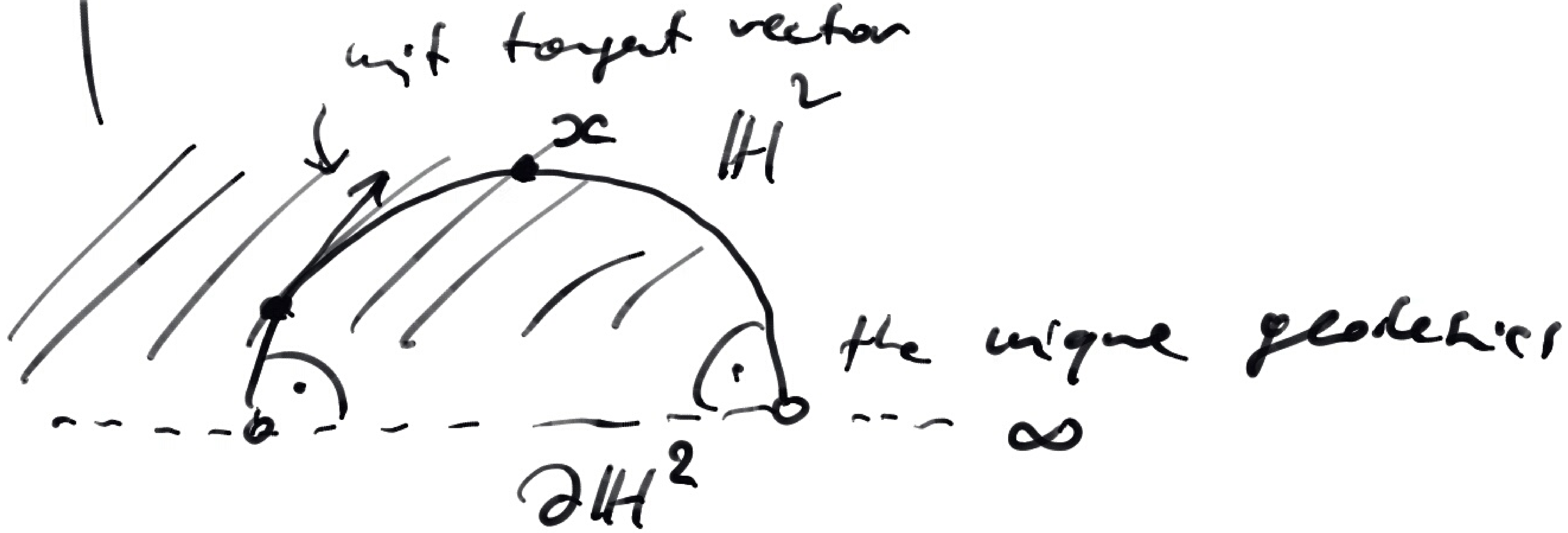
$\mathbb{H}^2 \times \mathbb{H}^2 \cong S\mathbb{H}^2$ (unit sphere bundle of $T\mathbb{H}^2$)

Γ acts isometrically on $\mathbb{H}^2$ and $S\mathbb{H}^2$

$\Rightarrow$ any $x \in \mathbb{H}^2$ defines a Γ-invariant Riemannian metric on $\partial\mathbb{H}^2$ which leads to a family of Dirac operators $(D_x)_{x\in\mathbb{H}^2}$

$$[(D)] \underset{x\in\mathbb{H}^2}{\in} KK^{\Gamma\ltimes\mathbb{H}^2}(C_0(\mathbb{H}^2\times\partial\mathbb{H}^2), C_0(\mathbb{H}^2)) \overset{\text{B.C.}}{\cong} KK^{\Gamma}(C(\partial\mathbb{H}^2), \mathbb{C})$$

$$\cong K^0(C(\partial\mathbb{H}^2)\rtimes\Gamma)$$



Theorem. [Higson–Kasparov] If G acts amenably on a commutative C^*-algebra B, then

$$p^*_{\underline{E}G} : KK^G(B, A) \longrightarrow KK^{G \ltimes \underline{E}G}(C_0(\underline{E}G, B), C_0(\underline{E}G, A))$$

is an isomorphism for every A.

Corollary. $A \rtimes_r G$ satisfies UCT.

$\exists (\mu_n)$ $\mu_n : X \to \mathrm{Prob}(G)$

s.t. $\lim\limits_{n\to\infty} \sup\limits_{x \in X} \| \gamma\mu_n(x) - \mu_n(\gamma x) \| = 0$

Remark. If G embeds uniformly in a Hilbert space, then $P^*_{\underline{E}G}$ is surjective. This implies the Novikov conjecture.

Construction of K-theory classes. [Baum-Connes]

G a loc. cpct group, X a smooth G-manifold, M a proper smooth G-manifold s.t. M/G cpct, $\pi : M \to X$ a G-equivariant K-oriented submersion. Then one can construct a family $(D_x)_{x \in X}$ of Dirac operators along the fibres of π.

This gives a class $\pi^! \in KK^G(C_0(M), C_0(X))$

Under the index map it is mapped into

$\mathrm{ind}_G(\pi^!) \in KK(\mathbb{C}, C(X) \rtimes G)$.

Theorem. [Baum-Connes] All K-theory classes for $C(X) \rtimes G$ arise in this way, if we allow twisting by G-equivariant vector bundles on M.